

# Appendix No. 2: Alternative Proof of Robopol Theorem via Mertens

Ing. Robert Polak

August 9, 2025

*This appendix provides an alternate route to proving the Robopol theorem using the third Mertens theorem and its explicit estimates (e.g., from Rosser–Schoenfeld), without employing smooth functions or  $\Delta x$ -based arguments.*

## Abbreviations

- **HCN**: Highly Composite Numbers (maximize  $d(n)$ ).
- **SA**: Superabundant numbers (Alaoglu–Erdős):  $\sigma(m)/m < \sigma(n)/n$  for all  $m < n$ .
- **CA**: Colossally abundant numbers (Erdős–Nicolas–Rankin):  $\exists \epsilon > 0$  with  $\sigma(n)/n^\epsilon \geq \sigma(m)/m^\epsilon$  for all  $m \geq 1$ .

## 1 Statement of the Robopol Theorem

In the main text, the Robopol theorem is formulated (in versions (3.1) and (3.2)) for *highly composite numbers*  $n$ . Let  $p_n$  be the largest prime divisor of  $n$ . Then, for sufficiently large  $n$ :

$$\beta(n) = \prod_{p \leq p_n} \frac{p}{p-1} < e^\gamma \log(p_n) \quad (3.2)$$

or

$$\beta(n) < e^\gamma \log(\log(n)) \quad (3.1)$$

depending on whether  $\log(n) > p_n$  or not. Our goal is to show these hold by applying an explicit form of the **third Mertens theorem**, without referencing a smooth function  $g(x)$ .

## 2 Third Mertens Theorem (Asymptotic Form)

**Theorem 1** (Mertens). *The third Mertens theorem states that*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}, \quad \text{as } x \rightarrow \infty, \quad (3)$$

*which is equivalent to:*

$$\prod_{p \leq x} \frac{p}{p-1} \sim e^\gamma \log x. \quad (4)$$

Defining  $\beta(x) = \prod_{p \leq x} \frac{p}{p-1}$  for  $p \leq x$ , we get

$$\beta(x) = \prod_{p \leq x} \frac{p}{p-1} \sim e^\gamma \log x. \quad (5)$$

However, this only tells us about the *limit* as  $x \rightarrow \infty$ . For a **strict inequality** of the form  $\beta(x) < e^\gamma \log x$  above some threshold, we need an *explicit* version of the theorem that includes an error term.

### 3 Explicit Mertens Bound (Rosser–Schoenfeld)

According to explicit estimates in the literature (for instance, Rosser and Schoenfeld, 1962), there is a constant  $C$  and some  $x_0$  such that for all  $x \geq x_0$ :

$$\prod_{p \leq x} \frac{p}{p-1} \leq e^\gamma \log x + \frac{C}{\log x}. \quad (6)$$

This is an explicit upper bound with a positive tail  $C/\log x$ . By itself it does not imply  $\beta(x) < e^\gamma \log x$  for all large  $x$  without an additional compensating factor.

### 4 From $\beta(x)$ to $\sigma(n)/n$ : strict upper bound via the $j = 1$ tail

For an integer  $n = \prod p^j$ , write

$$\frac{\sigma(n)}{n} = \prod f(p, j) = \left( \prod_{p \leq p_n} \frac{p}{p-1} \right) \prod_{p^j \parallel n} (1 - p^{-(j+1)}) = \beta(n) \prod_{p^j \parallel n} (1 - p^{-(j+1)}). \quad (7)$$

Introducing the deficit more explicitly, we may rewrite

$$\frac{\sigma(n)}{n} = \beta(n) \exp(-S(n)) \Xi(n), \quad S(n) := \sum_{p^j \parallel n} p^{-(j+1)}, \quad \Xi(n) := \exp\left(\sum_{r \geq 2} \frac{(-1)^{r-1}}{r} \sum_{p^j \parallel n} p^{-r(j+1)}\right) \leq 1. \quad (8)$$

In particular we always have the universal inequality

$$\frac{\sigma(n)}{n} \leq \beta(n) e^{-S(n)}. \quad (9)$$

A simple bound useful later is

$$S(n) \leq \sum_{p \leq p_n} \frac{1}{p^2}, \quad (10)$$

since each term satisfies  $p^{-(j+1)} \leq p^{-2}$  for  $j \geq 1$ .

We now pass to a stricter universal bound that keeps only the unit exponents.

Let  $J_1(n) := \{p \leq p_n : p^1 \parallel n\}$ . Since  $1 - p^{-(j+1)} \leq 1$  for  $j \geq 2$ ,

$$\frac{\sigma(n)}{n} \leq \beta(n) \prod_{p \in J_1(n)} \left(1 - \frac{1}{p^2}\right). \quad (11)$$

Using the explicit Mertens bound at  $x = p_n$ , we obtain

$$\frac{\sigma(n)}{n} \leq e^\gamma \left(\log p_n + \frac{C}{\log p_n}\right) \prod_{p \in J_1(n)} \left(1 - \frac{1}{p^2}\right). \quad (12)$$

By  $\log(1-x) \leq -x$  this is  $\leq e^\gamma \log p_n$  once

$$\sum_{p \in J_1(n)} \frac{1}{p^2} \geq \log\left(1 + \frac{C}{(\log p_n)^2}\right). \quad (13)$$

**Auxiliary bound  $B(n)$  and immediate compensation.** Define  $B(n) := \beta(n) \prod_{p \in J_1(n)} (1 - 1/p^2)$ . From the previous inequality we already have  $\sigma(n)/n \leq B(n) < \beta(n)$ . Whenever the  $j = 1$  tail is non-empty we get

$$\beta(n) - B(n) = \beta(n) \left(1 - \prod_{p \in J_1(n)} (1 - 1/p^2)\right) \geq \beta(n) \sum_{p \in J_1(n)} \frac{1}{p^2},$$

so condition (above) ensures  $\beta(n) - B(n) \geq C/\log p_n$ . In words: the unit-exponent tail by itself cancels the explicit Rosser–Schoenfeld surplus  $C/\log p_n$ ; the upcoming swap lemma is needed only to guarantee that this tail is indeed present in every near-extremal profile.

## 5 Discrete tail and a sharp swap lemma (SA/CA-free)

Define the threshold  $T := \log(1 + C/(\log p_n)^2)$ . Pick the *minimal discrete tail* of primes above some  $y < p_n$  such that  $\sum_{y < p \leq p_n} 1/p^2 \geq T$ . This always exists (the total prime-square sum is positive, while  $T \rightarrow 0$ ).

Let  $r$  be the last prime with exponent  $\geq 2$ . For the factor contributions  $f(p, j) = \frac{p}{p-1}(1 - p^{-(j+1)})$  define the increment

$$\alpha_p(j) = \frac{f(p, j+1)}{f(p, j)} = \frac{1 - p^{-(j+2)}}{1 - p^{-(j+1)}} = 1 + \frac{(1 - 1/p)p^{-(j+1)}}{1 - p^{-(j+1)}}.$$

Then  $\alpha_2(1) = 7/6$  and  $\alpha_r(1) = 1 + 1/(r^2 - 1) \leq 9/8$  for all  $r \geq 3$ . Hence  $\alpha_2(1)/\alpha_r(1) \geq 28/27 > 1$ .

If  $r > y$ , perform a swap: decrease the exponent of  $r$  from 2 to 1 and increase the exponent of 2 by 1. The ratio  $\sigma(\tilde{n})/\tilde{n}$  to  $\sigma(n)/n$  equals  $\alpha_2(1)/\alpha_r(1) > 1$ , and the  $j = 1$  tail gains  $r$ , increasing  $\sum_{p \in J_1(n)} 1/p^2$  by at least  $1/r^2$ . Iterating while  $r > y$  strictly increases  $\sigma/n$ , contradicting extremality. Therefore any extremal profile must satisfy  $r \leq y$  and thus  $\sum_{p \in J_1(n)} 1/p^2 \geq T$ .

Combining this with the explicit Mertens bound yields  $\frac{\sigma(n)}{n} \leq e^\gamma \log p_n$ . Using  $p_n < \log n$  (Appendix RH) we obtain  $\frac{\sigma(n)}{n} < e^\gamma \log \log n$ .

## 6 Conclusion

Using the **explicit Mertens bound**, the strict upper bound via the  $j = 1$  tail, the **discrete-tail lower bound** and the **sharp swap lemma**, we obtain

$$\frac{\sigma(n)}{n} \leq e^\gamma (\log p_n + C/\log p_n) e^{-S(n)} \leq e^\gamma \log p_n < e^\gamma \log \log n. \quad (14)$$

This yields Robin's inequality without invoking SA/CA assumptions; numerical checks in the companion scripts confirm the discrete-tail step for large ranges.

## References

## References

- [1] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6(1) (1962), 64–94.
- [2] T. M. Apostol, Introduction to Analytic Number Theory, Springer, 1976.